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On two-dimensional inviscid flow in a waterfall

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This paper is concerned with the two-dimensional flow in a free waterfall, falling under the influence of gravity, the fluid being considered to be incompressible and inviscid. A parameter ϵ, such that $2/\epsilon$ is the Froude number based on conditions far upstream, is defined and considered to be small. A flowline co-ordinate system is used to overcome the difficulty that the boundary geometry is not known in advance. An asymptotic expansion based on ϵ is constructed as an approximation valid upstream and near the edge, but singular far downstream. Another asymptotic expansion, based upon the thinness of the fall, is constructed as an approximation valid far downstream, but failing to satisfy the conditions upstream. The two expansions are then matched to give a solution covering the whole flow field. The shapes of the free streamlines are shown for a number of values of ϵ for which the solutions are seemingly valid.

1. Introduction

An inviscid, incompressible fluid flows over a horizontal bed until it falls over an edge under the influence of gravity. The flow is considered to be plane and steady. Far upstream the fluid is of depth h and has a uniform horizontal velocity U_0, and gravity is acting vertically downwards (see figure 1). The problem is one of finding the velocity potential Φ and the stream function Ψ as functions of position. Both Φ and Ψ must satisfy the Laplace equation subject to certain non-linear boundary conditions, namely zero pressure on the free streamlines and zero normal velocity on the bed. The basic non-dimensional parameter appearing in the problem is $\epsilon = 2gh/U_0^2$, and this is assumed to be small in most of this paper.

This problem involves a singular perturbation, the singularity occurring far downstream. As such, it lends itself to the technique of 'inner and outer expansions'. Kaplun & Lagerström (1957) and Erdélyi (1961) give a general account of this technique and also cite further references. In the present paper an expansion, which is derived to satisfy the conditions in that part of the flow which is not far downstream, will be known as the inner expansion, and the region in which it is valid, as the inner region. Similarly, the outer expansion satisfies the conditions far downstream, and is valid in the outer region.

The inner expansion is constructed by a perturbation scheme, in which all lengths are referred to h and all velocities to U_0; this scheme may be regarded as a perturbation for weak gravity. The first approximation is therefore a uniform horizontal stream. In the outer region, all lengths are referred to $U_0^2/2g$ and velocities again to U_0, and so in this region the perturbation may be regarded as one for small width of the fall. Here the first approximation, being hydraulic, is of the well-known parabolic form.

It is to be expected that the inner and outer regions overlap to some extent. By matching the inner and outer expansions in the overlap region, the unknown constants in the outer expansion are found, and the combined solutions then cover the whole flow field.

Southwell & Vaisey (1946) found a result for the case $\epsilon = 2$ by relaxation techniques, and their solution has been used for comparison purposes in figure 5. Keller & Weitz (1957) also found a solution in the outer region, though by an approach different from the one given in this paper. This solution was found to agree with ours to the first approximation.

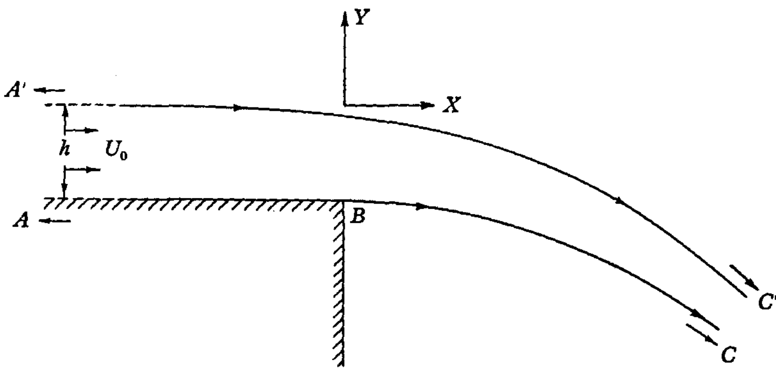

FIGURE 1. Notation.

2. Formulation

We denote the fluid velocity by $\mathbf{Q} = \nabla\Phi$, and consider a co-ordinate system $Z = X + iY$, in which the bed is described as $Y = -h$; $X \leqslant 0$. Gravity is acting in the direction of Y decreasing. The problem is to find the complex potential $F = \Phi + i\Psi$ satisfying $(\partial^2/\partial X^2 + \partial^2/\partial Y^2) F = 0$, subject to: (i) zero pressure on the free streamlines, (ii) zero normal velocity on the bed. The free streamlines are unknown in terms of X and Y, but are known in terms of Ψ. This suggests inverting the problem to one of finding Z as a function of F, that is, of finding Z satisfying $(\partial^2/\partial\Phi^2 + \partial^2/\partial\Psi^2) Z = 0$, subject to the same boundary conditions.

To find the boundary conditions explicitly, we make use of Bernoulli's equation:

$$P/\rho + \tfrac{1}{2}Q^2 + gY = \text{constant} = \tfrac{1}{2}U_0^2,$$

where the density ρ is constant throughout the fluid, and the constant on the right has been evaluated from the conditions far upstream on the upper free streamline.

We define non-dimensional variables by

$$p = P/\rho U_0^2; \quad q = |\mathbf{Q}|/U_0; \quad z = Z/h; \quad f = F/U_0 h;$$

and Bernoulli's equation becomes

$$2p + q^2 + \epsilon y = 1. \tag{2.1}$$

Therefore the boundary conditions are

$$\left.\begin{array}{llll}\text{(i)} & q^2 = 1-\epsilon y & \text{on} & \psi = 0, \quad \text{all } \phi; \\ \text{(ii)} & q^2 = 1-\epsilon y & \text{on} & \psi = -1, \quad \phi \geqslant 0; \\ \text{(iii)} & \operatorname{Im}\left(\dfrac{dz}{df}\right)^{-1} = 0 & \text{on} & \psi = -1, \quad \phi \leqslant 0.\end{array}\right\} \quad (2.2)$$

If we consider the problem to be in the complex f-plane, then the field equations are satisfied by any complex function $z(f)$. The problem is then, to find such a function $z(f)$ which satisfies (2.2).

3. The inner expansion

We pose that

$$z(f) = z_0(f) + \epsilon z_1(f) + \epsilon^2 z_2(f) + \dots. \quad (3.1)$$

To find the $z_n(f)$ we substitute (3.1) into (2.2) and, comparing coefficients of ϵ, obtain a sequence of linear problems in each of the $z_n(f)$ in turn.

$z_0(f)$ is simply the solution in the case when $\epsilon = 0$, and so $z_0(f) = f$.

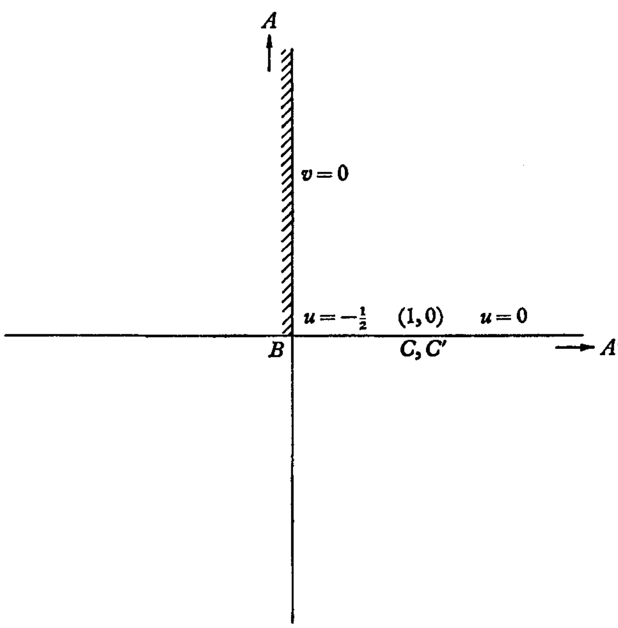

FIGURE 2. The complex s-plane, showing the boundary values of the first-order problem.

On substituting (3.1) into (2.2), and comparing first-order coefficients, we find that $x_{1\phi} = \frac{1}{2}\psi$ on the free streamlines, where the subscript ϕ denotes differentiation with respect to ϕ. We therefore seek $w_1 = u_1 + iv_1 = x_{1\phi} + iy_{1\phi}$, subject to

$$\left.\begin{array}{llll}\text{(i)} & u_1 = 0 & \text{on} & \psi = 0, \quad \text{all } \phi; \\ \text{(ii)} & u_1 = -\frac{1}{2} & \text{on} & \psi = -1, \quad \phi \geqslant 0; \\ \text{(iii)} & v_1 = 0 & \text{on} & \psi = -1, \quad \phi \leqslant 0.\end{array}\right\} \quad (3.2)$$

To solve this mixed boundary-value problem, we map the infinite strip, $0 \geqslant \psi \geqslant -1$, in the f-plane on to the upper right-hand quadrant of the $s = s_1 + is_2$ plane by the mapping: $s = \sqrt{(1+e^{-\pi f})}$. The boundary conditions then are as given by (3·3), and as shown in figure 2.

$$\left.\begin{array}{llll} \text{(i)} & u_1 = 0 & \text{on} & s_2 = 0, \quad s_1 \geqslant 1; \\ \text{(ii)} & u_1 = -\tfrac{1}{2} & \text{on} & s_2 = 0, \quad 0 \leqslant s_1 \leqslant 1; \\ \text{(iii)} & v_1 = 0 & \text{on} & s_1 = 0, \quad s_2 \geqslant 0. \end{array}\right\} \tag{3.3}$$

This problem is familiar, in that it is analogous to that of finding the complex potential of an inviscid flow, covering the entire plane, with a pair of vortices situated at (1,0) and (− 1, 0). The solution is well known:

$$w_1(s) = \frac{i}{2\pi}\log\left(\frac{s-1}{s+1}\right). \tag{3.4}$$

However, it will be more helpful to solve a more general mixed boundary-value problem, as this more general solution may be used in the higher-order problems.

Consider a complex function $w_n = u_n + iv_n$, analytic in $s_1 \geqslant 0$, $s_2 > 0$, with u_n prescribed on the positive real axis. Following Woods's (1961) account, we assume that

(i) $v_n = 0$ on $s_1 = 0$, $s_2 \geqslant 0$;

(ii) $w_n(s) \sim O(s^{-1})$ as $|s| \uparrow \infty$;

(iii) $w_n(s)$ is integrable in the ordinary (Riemann) sense on any finite arc of the positive real axis. (Unlike Woods we do not allow w_n to have singularities of the Cauchy type.) (3.5)

If we consider the problem to be in the whole of the upper half plane, with u_n now also prescribed on the negative real axis, then the solution is well known,

$$w_n(s) = -\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{u_n(\sigma)}{\sigma - s}\,d\sigma. \tag{3.6}$$

To ensure that $v_n = 0$ on the positive imaginary axis, we have that

$$u_n(\sigma) = u_n(-\sigma),$$

and using this
$$w_n(s) = \frac{i}{\pi}\int_0^{\infty} u_n(\sigma)\left\{\frac{1}{s+\sigma} + \frac{1}{s-\sigma}\right\}d\sigma. \tag{3.7}$$

We return to the first-order problem. The boundary conditions satisfy (3.5), and so using this method we recover (3.4).

By the restriction (3.5, (iii)) we have excluded terms in $w_n(s)$ of the form

$$i\left(\frac{1}{s-1} + \frac{1}{s+1}\right),$$

which may be added to any solution without violating the boundary conditions except at the singular point $s = 1$. This is because we accept only the weakest possible singularity for $s \to 1$, a policy justified later by the matching procedure. Therefore the solution to the first-order problem is given by (3.4), which in terms of the original variables, becomes

$$z_{1f} = \frac{i}{2\pi}\log\left\{\frac{(1+e^{-\pi f})^{\frac{1}{2}} - 1}{(1+e^{-\pi f})^{\frac{1}{2}} + 1}\right\}. \tag{3.8}$$

When we come to the matching procedure, we will require an expression for $z_1(f)$ as $f \uparrow \infty$; this is then from (3.8)

$$z_1(f) \sim -\frac{i}{4} f^2 - \left(\frac{i}{\pi} \log 2\right) f + \text{const.} + O(e^{-\pi f}). \tag{3.9}$$

We now turn our attention to the second-order coefficient, z_2. On inserting (3.1) into (2.2), and comparing second-order terms, we have that

$$x_{2\phi} = \tfrac{1}{2}(y_1 + 3x_{1\phi}^2 - y_{1\phi}^2)$$

on the free streamlines. We therefore seek $w_2 = u_2 + iv_2 = x_{2\phi} + iy_{2\phi}$, subject to

$$\left.\begin{array}{llll} \text{(i)} & u_2 = \tfrac{1}{2}(y_1 - y_{1\phi}^2) & \text{on} \quad \psi = 0, & \text{all } \phi; \\ \text{(ii)} & u_2 = \tfrac{1}{2}(y_1 - y_{1\phi}^2) + \tfrac{3}{8} & \text{on} \quad \psi = -1, & \phi \geqslant 0; \\ \text{(iii)} & v_2 = 0 & \text{on} \quad \psi = -1, & \phi \leqslant 0. \end{array}\right\} \tag{3.10}$$

Mapping the f-plane onto the s-plane, we find that on $s_2 = 0$, u_2 has a finite discontinuity, and singularities of the nature $\log^2|s_1 - 1|$ and $\log|s_1 - 1|$ at $s_1 = 1$, but has no singularities elsewhere. Hence u_2 satisfies (3.5, (iii)). Also

$$u_2 = O(s^{-2}) \quad \text{as} \quad |s| \uparrow \infty,$$

and so all the conditions in (3.5) are satisfied. Therefore, using (3.7), the solution is given by

$$w_2(s) = -\frac{3i}{8\pi} \log\left(\frac{s-1}{s+1}\right) + \frac{i}{\pi}\int_0^\infty G(\sigma)\left\{\frac{1}{s-\sigma} + \frac{1}{s+\sigma}\right\} d\sigma, \tag{3.11}$$

where $G(\sigma) = \frac{1}{2}(y_1 - y_{1\phi}^2)$ on $s = \sigma$, σ real. The behaviour of $G(\sigma)$ near $\sigma = 1$, is given by

$$G(\sigma) = -\frac{1}{4\pi^2}\log^2|\sigma - 1| + \frac{1}{2\pi^2}\log 2 \log|\sigma - 1| + \tfrac{1}{8} H(1-\sigma) + J(\sigma),$$

where H is the Heaviside unit function, and $J(\sigma)$ is regular at $\sigma = 1$. To remove this singularity from within the integral, we define the complex function $\gamma(s) = \alpha(s_1, s_2) + i\beta(s_1, s_2)$ by

$$\gamma(s) = -\frac{1}{4\pi^2}\log^2\left(\frac{s-1}{s+1}\right) + \frac{i}{8\pi}\log\left(\frac{s-1}{s+1}\right) - \frac{1}{4\pi^2}(s-1)\log(s-1)$$
$$+ \frac{1}{4\pi^2}(s+1)\log(s+1) - \frac{1}{2\pi^2}\log(s+i) - \frac{1}{2\pi^2}. \tag{3.12}$$

The function $\gamma(s)$ satisfies the conditions (3.5), and $G(\sigma) - \alpha(\sigma, 0) = \Omega(\sigma)$, where $\Omega(\sigma)$ and $d\Omega(\sigma)/d\sigma$ are continuous in $0 \leqslant \sigma \leqslant \infty$. From (3.7)

$$\gamma(s) = \frac{i}{\pi}\int_0^\infty \alpha(\sigma, 0)\left\{\frac{1}{s-\sigma} + \frac{1}{s+\sigma}\right\} d\sigma. \tag{3.13}$$

Subtracting (3.13) from (3.11) we have

$$w_2(s) = \frac{-3i}{8\pi}\log\left(\frac{s-1}{s+1}\right) + \gamma(s) + \frac{i}{\pi}\int_0^\infty \Omega(\sigma)\left\{\frac{1}{s-\sigma} + \frac{1}{s+\sigma}\right\} d\sigma. \tag{3.14}$$

The behaviour of $z_2(f)$ as $f \uparrow \infty$, is then

$$z_2(f) \sim -\tfrac{1}{12}f^3 + \frac{1}{2}\left(\frac{i}{4} - \frac{1}{\pi}\log 2\right)f^2 - \left(\frac{1}{\pi^2}\log^2 2 + \frac{1}{2\pi^2} + \frac{i}{8\pi} - \frac{i}{\pi}\log 2\right)f + \text{const.} + O(e^{-\pi f}). \tag{3.15}$$

It is worth noting here that on the lower streamline near the edge

$$z_1 \sim -\tfrac{1}{2}\phi - i\frac{2}{3\sqrt{\pi}}\phi^{\frac{3}{2}} + O(\phi^{\frac{5}{2}}),$$

and

$$z_2 \sim A_0\phi + i\frac{2A_1}{3}\phi^{\frac{3}{2}} + O(\phi^{\frac{5}{2}}),$$

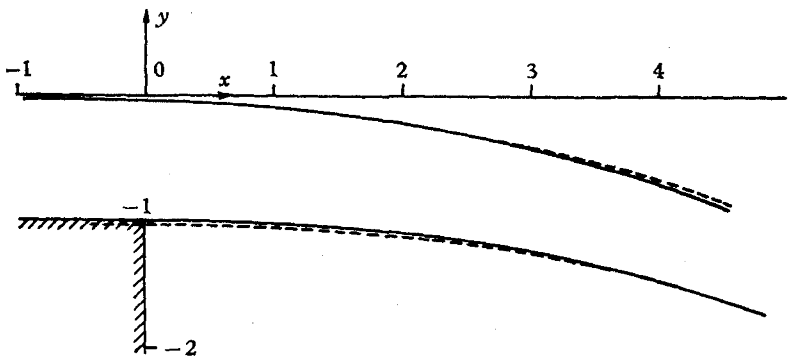

FIGURE 3. Case of $\epsilon = 0{\cdot}1$; ———, inner solution; - - - -, outer solution.

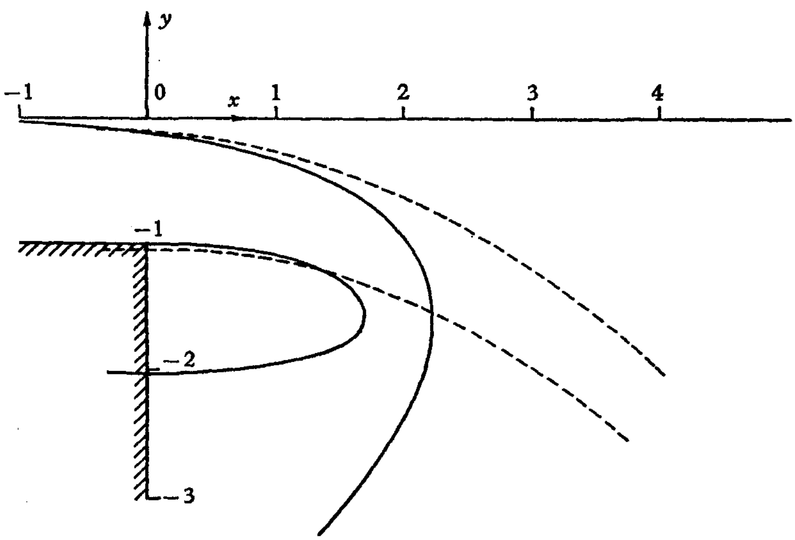

FIGURE 4. Case of $\epsilon = 0{\cdot}5$; ——, inner solution, - - - -, outer solution.

where A_0, A_1 are real finite constants. In both these expressions, the leading singular terms are of order $\phi^{\frac{3}{2}}$. This shows that the singularity in the first-order term does not give rise to a more singular term in the second-order expression. It would appear, then, that at the edge, $z(f)$ has no worse a singularity than that contained in $z_1(f)$.

The functions x_1, x_2, y_1, y_2, have been evaluated numerically for the upper and lower free streamlines in the range $-5 \leqslant \phi \leqslant +5$, and the results have been used in the construction of the figures 3 and 4. The higher-order terms in (3.1) could be derived in a similar manner, but we terminate the inner expansion after the second-order term.

4. The outer expansion

Defining the complex velocity by $qe^{-i\theta}$, we know that $qe^{-i\theta} = \text{fn}(\phi + i\psi; \epsilon)$, but we do not know the manner in which ϵ enters this function for large values of ϕ. However, if we take $U_0^2/2g$ as reference length, and U_0 as reference velocity, this makes the width of the fall of order ϵ. That is, if ψ^+ is the new non-dimensional stream function, then the flow is bounded by the streamlines $\psi^+ = 0$, $\psi^+ = -\epsilon$. This narrowness is useful so long as $\partial/\partial\psi^+ \sim O(1)$, for then we may assume little change across the fall. We have from the boundary conditions on the free streamlines

$$[q^2]_{\psi=-1}^{\psi=0} = -\epsilon[y(\phi,\psi)]_{\psi=-1}^{\psi=0} \quad (\phi > 0). \tag{4.1}$$

This indicates that $\partial/\partial\psi^+ \sim O(1)$ far downstream, and we therefore adopt $U_0^2/2g$ and U_0 as the reference length and velocity in this region. Then $z^+ = z^+(f^+;\epsilon)$, where

$$z^+ = \frac{2gZ}{U_0^2} = \epsilon z \quad \text{and} \quad f^+ = \frac{2gF}{U_0} = \epsilon f = \phi^+ + i\epsilon\psi.$$

We define the outer limit to be

$$\epsilon \downarrow 0, \quad \text{with} \quad \phi^+, \psi \quad \text{fixed} \quad \phi^+ > 0; \quad \text{applied to} \quad z^+(\phi^+ + i\epsilon\psi;\epsilon),$$

whereas the inner limit was

$$\epsilon \downarrow 0, \quad \text{with} \quad \phi, \psi \quad \text{fixed,} \quad \phi < \infty; \quad \text{applied to} \quad z(\phi + i\psi;\epsilon).$$

It will be noted that ϵ does not appear in the boundary conditions, but in the actual boundary $\psi^+ = 0$, $\psi^+ = -\epsilon$.

The expression $z^+ = z^+(f^+;\epsilon)$ suggests that we could expand z^+ in a power series of the form

$$z^+ = z_0^+(\phi^+ + i\epsilon\psi) + \epsilon z_1^+(\phi^+ + i\epsilon\psi) + \ldots, \tag{4.2}$$

and with direct substitution of (4.2) into the boundary conditions; $q^2 = 1 - y^+$ on $\psi^+ = 0$ and $\psi^+ = -\epsilon$, we would obtain a sequence of non-linear, ordinary differential equations for $x_n^+(\phi^+, 0)$ and $y_n^+(\phi^+, 0)$, which could be solved.

However, we approach the problem from a different viewpoint. The following derivation is more satisfactory in that it is simpler, sheds more light on the physical problem, and leads to a series valid not only under the outer limit previously defined, but also under two other limits.

First, we change to a less cumbersome notation, writing

$$z^+ = \zeta = \xi + i\eta; \quad f^+ = \tau; \quad \phi^+ = \sigma.$$

We have then that $\log q - i\theta = \text{fn}(\tau;\epsilon)$, and therefore, by the Cauchy–Riemann relations,

$$\epsilon q^{-1} q_\sigma = -\theta_\psi, \tag{4.3}$$

$$q^{-1} q_\psi = \epsilon\theta_\sigma, \tag{4.4}$$

Also from the definitions of ϕ and ψ,

$$d\xi = q^{-1}(\cos\theta\, d\sigma - \epsilon \sin\theta\, d\psi), \tag{4.5}$$

$$d\eta = q^{-1}(\sin\theta\, d\sigma + \epsilon \cos\theta\, d\psi). \tag{4.6}$$

The boundary conditions are

$$q^2 = 1 - \eta \quad \text{on} \quad \psi = 0, \quad \psi = -1. \tag{4.7}$$

By considering momentum flux in the ξ-direction, it can be shown that

$$\int_{-1}^{0} (p/q + q)\cos\theta\, d\psi = \text{const.} = 1 + \tfrac{1}{4}\epsilon = E, \tag{4.8}$$

where the flow conditions far upstream have been used to evaluate the constant on the right, and p is given by

$$p = \tfrac{1}{2}(1 - \eta - q^2). \tag{4.9}$$

In terms of the variable ψ, the width of the fall is $O(1)$, and derivatives with respect to ψ are $O(\epsilon)$, and so we may take as a first approximation that q and θ are independent of ψ, and also that $q^2 \gg p$. Then from (4.8) we have

$$q_0 \sim E \sec\theta_0, \tag{4.10}$$

where the subscript '0' denotes the value taken on $\psi = 0$. Also from (4.7) and (4.6), $q_0^2 = 1 - \eta_0$ and

$$\left(\frac{d\eta}{d\sigma}\right)_0 = q_0^{-1} \sin\theta_0,$$

which, with (4.10) give

$$\tfrac{2}{3}(q_0^2 - E^2)^{\frac{3}{2}} + 2E^2(q_0^2 - E^2)^{\frac{1}{2}} \sim \sigma - \Lambda(\epsilon),$$

where $\Lambda(\epsilon)$ is a constant of integration.

We define $\lambda(\sigma, \epsilon)$ by $\theta_0 = -\lambda$; then

$$\sigma - \Lambda(\epsilon) \sim 2E^3(\tan\lambda + \tfrac{1}{3}\tan^3\lambda). \tag{4.11}$$

We may now take λ, rather than σ, to be the independent variable, and express all other quantities in terms of λ, equation (4.11) providing the link with the original variable. In this case we then have

$$q_0 \sim E \sec\lambda, \tag{4.12}$$

$$\eta_0 \sim -E^2 \sec^2\lambda + 1, \tag{4.13}$$

$$\xi_0 \sim \Delta(\epsilon) + 2E^2 \tan\lambda. \tag{4.14}$$

(4.14) and (4.13) clearly show the parabolic form of the fall, to the first approximation. (4.14) was constructed by using (4.5).

We express q, η, ξ and θ in the form of Taylor series about $\psi = 0$, viz.;

$$q = q_0 + (q_\psi)_0 \cdot \psi + \dots.$$

Using (4.3)–(4.6), we can show that

$$q \sim E\sec\lambda - (\epsilon\psi \cos^3\lambda)/2E^2, \tag{4.15}$$

$$\theta \sim -\lambda - (\epsilon\psi \cos^3\lambda \sin\lambda)/2E^3, \tag{4.16}$$

$$\eta \sim -E^2\sec^2\lambda + 1 + (\epsilon\psi\cos^2\lambda)/E, \tag{4.17}$$

$$\xi \sim \Delta(\epsilon) + 2E^2\tan\lambda + (\epsilon\psi \sin\lambda\cos\lambda)/E. \tag{4.18}$$

To find a second approximation, we put $q_0 = E\sec\lambda + q_1$, and neglect all terms of $O(q_1^2)$, so that $\eta_0 = -E^2\sec^2\lambda + 1 - 2E\sec\lambda q_1$. On substituting these values into (4.15) and (4.17), and then using the new values of (4.15) and (4.17) in (4.8), we find that

$$q_1 = -(\epsilon\cos\lambda)/4E^2.$$

It is then easily shown that the full second-order approximations are

$$q \sim E\sec\lambda - (\epsilon\cos\lambda)|4E^2 - (\epsilon\psi\cos^3\lambda)|2E^2, \tag{4.19}$$

$$\theta \sim -\lambda - (\epsilon\psi\cos^3\lambda\sin\lambda)|2E^3, \tag{4.20}$$

$$\eta \sim -E^2\sec^2\lambda + 1 + \epsilon|2E + (\epsilon\psi\cos^2\lambda)|E, \tag{4.21}$$

$$\xi \sim \Delta(\epsilon) + 2E^2\tan\lambda + (\epsilon\psi\sin\lambda\cos\lambda)|E, \tag{4.22}$$

and

$$\frac{d\lambda}{d\sigma} \sim \cos^4\lambda/[2E^3(1-\epsilon\cos^2\lambda|4E^3)], \tag{4.23}$$

so that

$$\sigma - \Lambda(\epsilon) \sim 2E^3(\tan\lambda + \tfrac{1}{3}\tan^3\lambda) - \tfrac{1}{2}(\epsilon\tan\lambda). \tag{4.24}$$

The equations (4.19)–(4.22) have the appearance of asymptotic expansions under three different limits, namely

(i) $\epsilon \downarrow 0$ with ψ, λ fixed, and $\lambda > 0$,

(ii) $\epsilon \uparrow \infty$ with ψ, λ fixed, and $\lambda > 0$,

(iii) $\lambda \uparrow \frac{1}{2}\pi$ with ϵ, ψ fixed, and $\epsilon \geqslant 0$.

It should be noted that in the case of the limit (ii) $E = 1 + \frac{1}{4}\epsilon \sim \frac{1}{4}\epsilon$, and so $\epsilon/E \ll E$. However only in the limit (i) can the unknown constants, $\Delta(\epsilon)$ and $\Lambda(\epsilon)$, be determined by matching.

5. The matching procedure

We consider the limiting process, $\epsilon \downarrow 0$ for $f = m(\epsilon)f_m$, with f_m fixed, and $1 \ll m(\epsilon) \ll \epsilon^{-1}$, where the notation $a(\epsilon) \ll b(\epsilon)$ means $a/b \downarrow 0$, as $\epsilon \downarrow 0$; $a, b \geqslant 0$. f_m is called an intermediate variable because;

$$f = m(\epsilon)f_m \uparrow \infty, \text{ as } \epsilon \downarrow 0 \text{ with } f_m \text{ fixed,}$$

and

$$\tau = \epsilon m(\epsilon)f_m \downarrow 0, \text{ as } \epsilon \downarrow 0 \text{ with } f_m \text{ fixed.}$$

We now assume that the set of intermediate order functions $m(\epsilon)$ defines an overlap region in which the inner expansion, the outer expansion and the exact solution are all asymptotically equal. Therefore we express the inner expansion, in terms of the intermediate variables, for $f \uparrow \infty$, and the outer expansion, also in terms of the intermediate variable, for $\tau \downarrow 0$, and compare the two resulting expansions.

We have, from § 3, the result that for $f \uparrow \infty$

$$z = m(\epsilon)f_m + \epsilon\left[-\frac{i}{4}m^2(\epsilon)f_m^2 - \frac{i}{\pi}\log 2m(\epsilon)f_m + \text{const}\right] + O(\epsilon^2 m^3(\epsilon)). \tag{5.1}$$

If in (4.21) and (4.22), we express λ in a double series in ϵ and σ, and making use of (4.23), we can put the outer expansion into the form,

$$\begin{aligned} z = {} & \epsilon^{-1}[(\Delta_0 + 2\tan c_0 + i\tan^2 c_0) + \epsilon m(\epsilon)f_m\cos^2 c_0(1 - i\tan^2 c_0) \\ & + \tfrac{1}{2}\epsilon^2 m^2(\epsilon)f_m^2(-\cos^3 c_0\tan c_0 + \tfrac{1}{2}i\cos^4 c_0(2\sin^2 c_0 - 1))] + O(\epsilon^2 m^3(\epsilon)) \\ & + [(\Delta_1 + \tfrac{1}{2}\tan c_0 - \tfrac{1}{2}\sin c_0\cos c_0 + 2a_1\cos^2 c_0 + i(\tfrac{1}{2}\sin^2 c_0 - 2a_1\sin c_0\cos c_0))] \\ & + \epsilon m(\epsilon)f_m[(\tfrac{1}{2}\sec^2 c_0 + \tfrac{1}{2}\sin^2 c_0 - \tfrac{1}{2}\cos^2 c_0 - 4a_1\cos c_0\sin c_0) + i(2\sin c_0\cos c_0 \\ & - 2a_1\cos 2c_0)]\,\tfrac{1}{2}\cos^4 c_0 + O(\epsilon^2 m^2) + O(\epsilon), \end{aligned} \tag{5.2}$$

where c_0 is the value of λ for $\epsilon = 0$ and $\sigma = 0$, and $\Delta(\epsilon) = \Delta_0 + \epsilon\Delta_1 + \dots$. a_1 is a constant to be determined and is related to Λ_1, where

$$\Lambda(\epsilon) = \Lambda_0 + \epsilon\Lambda_1 + \dots.$$

On comparing (5.1) and (5.2), we find that, from first-order terms

$$\Delta_0 + 2\tan c_0 = 0, \tag{5.3}$$

$$\tan^2 c_0 = 0, \tag{5.4}$$

and we can thus deduce from these that

$$c_0 = 0, \quad \Delta_0 = 0.$$

We can also deduce, from the size of the terms we have neglected, that

$$1 \ll m(\epsilon) \ll \epsilon^{-1},$$

and so for matching to one term, the overlap region is defined by

$$f = m(\epsilon) f_m; \quad 1 \ll m(\epsilon) \ll \epsilon^{-1}, \quad 0 < f_m < \infty.$$

On putting c_0 and Δ_0 to zero in (5.2), we have

$$z = m(\epsilon) f_m - \frac{i}{4}\epsilon m^2(\epsilon) f_m^2 + (\Delta_1 + 2a_1) - ia_1 \epsilon m(\epsilon) f_m + O(\epsilon^2 m^3(\epsilon)) + O(\epsilon^2 m^2(\epsilon)) + O(\epsilon). \tag{5.5}$$

On comparing (5.5) and (5.1), the first two terms in each are the same, and from the other terms we have that

$$\Delta_1 + 2a_1 = 0, \tag{5.6}$$

$$a_1 = \frac{1}{\pi}\log 2, \tag{5.7}$$

and so

$$\Delta_1 = -\frac{2}{\pi}\log 2. \tag{5.8}$$

Also from the neglected terms, we can deduce that

$$1 \ll m(\epsilon) \ll \epsilon^{-\frac{1}{2}}.$$

Therefore, for matching to two terms, the overlap region is defined by

$$f = m(\epsilon) f_m, \quad 1 \ll m(\epsilon) \ll \epsilon^{-\frac{1}{2}}, \quad 0 < f_m < \infty.$$

From (5.7) and (4.24), we find that

$$\Lambda_0 = 0, \quad \Lambda_1 = -\frac{2}{\pi}\log 2.$$

Therefore by matching we have found that

$$\Delta(\epsilon) = -\frac{2}{\pi}\log 2 \,.\, \epsilon + O(\epsilon^2),$$

and

$$\Lambda(\epsilon) = -\frac{2}{\pi}\log 2 \,.\, \epsilon + O(\epsilon^2).$$

Also, the fact that the two expansions do have the same asymptotic form in the overlap region, provides a strong indication that our assumptions, as to the form the expansions should take, were correct.

The determination of the constants in the outer solution provides a complete solution covering the whole flow field. Figures 3 and 4 show this solution, for the upper and lower streamlines, in the cases when $\epsilon = 0{\cdot}1$ and $\epsilon = 0{\cdot}5$. In the latter case, the inner solution displays a tendency towards a reversal

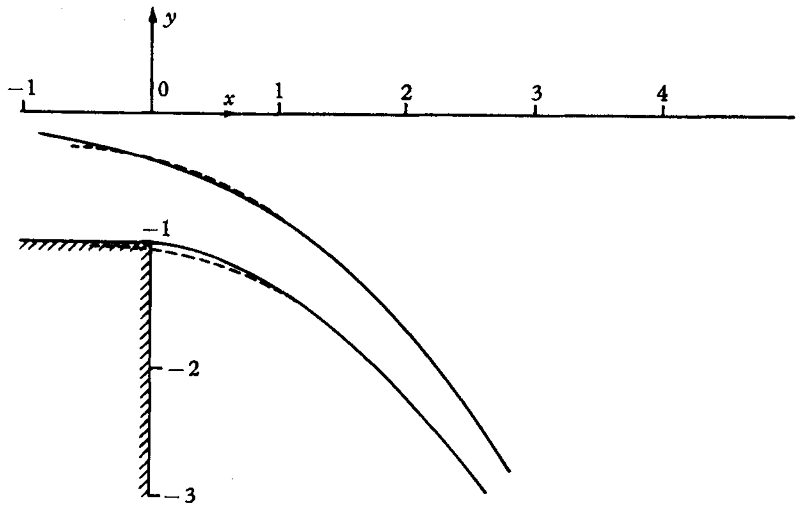

FIGURE 5. Case of $\epsilon = 2{\cdot}0$; comparison between our outer solution and the solution of Southwell & Vaisey (1946). ----, Outer solution; ———, Southwell & Vaisey solution.

in the direction of the flow, a tendency which becomes more severe with increasing ϵ. In figure 5, the outer solution is shown to be in close agreement with the Southwell & Vaisey solution for $\epsilon = 2$, though, for this case, the inner solution is such that it does not coincide with the outer solution before reversal occurs.

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